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Energy stability of modulated circular Couette flow

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The stability of circular Couette flow when the outer cylinder is at rest and the inner is modulated both with and without a mean shear is examined in the narrow-gap limit. Disturbances are assumed to be axisymmetric. Two criteria are used to determine conditions for stability; the first requires that the motion be strongly stable, the second only that disturbances of arbitrary initial energy decay from cycle to cycle. The behaviour of critical parameters as a function of frequency is similar for the linear and the energy analysis. The range of Reynolds numbers bounded above by certain instability and below by conditional non-linear stability is enlarged by modulation.

1. Introduction

We are concerned with the stability of motion between two infinitely long coaxial cylinders when the outer cylinder is at rest and the inner has angular velocity

$$\Omega_1 = \Omega_m + \Omega_p \cos \omega' t'. \tag{1}$$

The problem with Ω_m and Ω_p both non-zero has been studied experimentally by Donnelly (1964). He used an apparatus which produced an electric current proportional to the radial perturbation to the flow. In the absence of instability the signal should be constant. The current was integrated over a cycle of modulation; we shall refer to this quantity as the amplitude. For steady flow, bifurcation is known to be one-sided, and the amplitude of secondary motion is found to increase as

$$\text{amplitude} \sim (\mathscr{R} - \mathscr{R}_0^L)^{\frac{1}{2}} \tag{2}$$

(Kirchgässner & Sorger 1969; Davey 1962). Here $\mathscr{R}$ is the Reynolds number and $\mathscr{R}_0^L$ the critical Reynolds number according to linear theory. Donnelly assumed that this would also hold for bifurcation of the modulated problem, and considered that the flow became unstable when the amplitude first began to increase rapidly in approximate correspondence with (2). On this basis he concluded that modulation stabilized the flow for $\gamma \leqslant 1{\cdot}0$ (i.e. if Ω_0 is the critical speed in the absence of modulation, when $\Omega_p = 0$, then for $\Omega_p > 0$ instability occurs for $\Omega_m > \Omega_0$). The frequency parameter γ is defined by

$$\gamma = (\omega' d^2/2\nu)^{\frac{1}{2}}, \tag{3}$$

where d is the gap width and ν the kinematic viscosity.

Analysis of Donnelly's data suggests a trend towards destabilization for $\gamma > 1$. There is some ambiguity in Donnelly's interpretation of his data. He observed departures from the equilibrium configuration as soon as

$$\Omega_m + \Omega_p \geqslant \Omega_0 \tag{4}$$

but considered these to be 'transient vortices' and not a true manifestation of instability as they did not amplify according to relation (2).

Thompson (1968) presented experimental data from visual observation of the onset of instability for modulated flow with zero mean, i.e. $\Omega_m = 0$. This configuration may be considered more stable than steady flow, but in the limit of zero frequency the data are asymptotic to Ω_0, i.e. as $\gamma \to 0$ the flow becomes unstable as soon as

$$\Omega_p \geqslant \Omega_0. \tag{5}$$

He remarks that, if the flow has non-zero mean, instability will be observed at low frequencies whenever (4) is satisfied, while at higher frequencies no instability is apparent until

$$\Omega_m \geqslant \Omega_0. \tag{6}$$

This would be consistent with Donnelly's findings if we supposed that the 'transient vortices' were indeed true secondary motion, i.e. modulation is destabilizing at low frequencies. We note that Donnelly measured the amplitude of secondary motion as a function of Reynolds number. However, the transient vortices, which appeared whenever (4) was satisfied, are found, by comparison with the linear theory of the modulated problem (cf. Hall 1975; Riley & Laurence 1976), to be subcritical. Donnelly measured, therefore, the amplitude of a branch of secondary motion a portion of which was subcritical. There is no reason to suppose that the branch is tangential to the manifold of the linear solution, and consequently no reason to expect the amplitude to behave as (2).

The linear stability of the problem studied experimentally by Donnelly has been examined by Hall (1975) and Riley & Laurence (1976) in the narrow-gap limit for axisymmetric disturbances. Hall solved the problem by perturbation techniques in three asymptotic limits:

$$\text{(i)}\ \gamma \to 0 \quad \text{with} \quad \epsilon = k\gamma,$$

where $\epsilon = \Omega_p/\Omega_m$ is the modulation amplitude and k is a constant,

$$\text{(ii)}\ \gamma \gg 1 \quad \text{with} \quad \epsilon < \infty$$

and

$$\text{(iii)}\ \epsilon \ll 1 \quad \text{for arbitrary}\ \gamma.$$

He concluded that modulation is destabilizing. Riley & Laurence solved the problem using Galerkin's method and Floquet theory. They found that modulation was destabilizing in general but led to weak stabilization at larger amplitude ratios ($\epsilon \geqslant 2{\cdot}0$) and intermediate frequencies associated primarily with a half-frequency response. Their results are in excellent agreement with cases (i) and (iii) of Hall and with case (ii) when $\gamma \to \infty$.

A comparison of the linear results of Hall (1975) and Riley & Laurence (1976) with the data of Donnelly (1964) and Thompson (1968), as interpreted above, suggests that theory and experiment are in agreement at intermediate and higher frequencies, say $\gamma \geqslant 2{\cdot}0$, but at lower frequencies, $\gamma \leqslant 1{\cdot}5$, instability appears to be subcritical. (There is, however, a paucity of data for comparison.) Therefore we choose to examine the problem using energy theory, not only to determine whether this might not be a more reliable predictive technique at lower frequencies, but also to evaluate the region open to potential subcritical instability for the frequency range of interest.

We consider the magnitude of $\mathscr{R}^L - \mathscr{R}^E$ to provide some qualitative indication of the likelihood of subcritical instability. $\mathscr{R}^L$ and $\mathscr{R}^E$ are the critical Reynolds numbers from linear and energy theory respectively. Evidently, if $\mathscr{R}^L = \mathscr{R}^E$ subcritical motion is not possible. In general, it is not possible to make the relation concrete. If, for example, bifurcation is one-sided (for $\mathscr{R} > \mathscr{R}^L$) subcritical instability is not expected.† If, however, bifurcation is two-sided,‡ subcritical motion is not only possible but probable. For the stability of steady flows, systems for which bifurcation is one-sided have $\mathscr{R}^L - \mathscr{R}^E$ 'small', e.g. the Taylor problem. Motions which demonstrate subcritical instability and whose bifurcation is two-sided have $\mathscr{R}^L - \mathscr{R}^E$ 'large', e.g. plane Poiseuille flows. This relation is not precise, however. For two-sided bifurcations the subcritical branch is known to be unstable both for steady primary flows (Joseph & Sattinger 1972) and for time-periodic primary flows (Joseph 1972). This branch, however, is bounded below by $\mathscr{R}^E$, and passes through a positive minimum, where stability is regained. The energy bound, therefore, may be of considerable interest in this case.

In the course of this work we shall compare the results of linear and energy theory. We shall say that energy (or linear) theory predicts that modulation is destabilizing if $\mathscr{R}^E$ (or $\mathscr{R}^L$) is reduced below the corresponding value $\mathscr{R}_0^E$ (or $\mathscr{R}_0^L$) for the steady problem. This is potentially misleading since $\mathscr{R}^E$ provides only a sufficient condition for stability, and in general provides no information on instability. It is, however, convenient usage and no confusion should arise.

The energy method has previously been applied to modulated circular Couette flow in the narrow-gap limit for axisymmetric disturbances by Conrad & Criminale (1965). They treat the system studied by Donnelly as one of many cases. Several aspects of their work appear questionable. They find weak stabilization for small amplitude ratios in agreement with the data of Donnelly, but energy theory certainly predicts that modulation is destabilizing in the low frequency limit (see § 3.2). They find for modulation of the outer cylinder with non-zero mean (the inner being at rest) that a critical Reynolds number, based on the maximum velocity of the outer cylinder, tends to zero as the frequency

† There may of course be branches corresponding to finite amplitude secondary motion as distinct from that tangential to the linear solution. These could be subcritical. However, the existence or non-existence of such branches has not been proved.

‡ In the neighbourhood of $\mathscr{R}^L$, the relationship between the Reynolds number $\mathscr{R}$ and some measure ζ of the amplitude of the difference motion may be expressed as a power series

$$\mathscr{R} = \mathscr{R}_0 + \zeta\mathscr{R}_1 + \zeta^2\mathscr{R}_2 + \ldots$$

(Joseph & Sattinger 1972). If $\mathscr{R}_1 = 0$ then either $\mathscr{R} < \mathscr{R}^L$ or $\mathscr{R} > \mathscr{R}^L$ for all ζ close to zero. In this case the bifurcation is termed one-sided. Normally the supercritical branch exists and is stable. If, however, $\mathscr{R}_1 \neq 0$, the Reynolds number satisfies both $\mathscr{R} < \mathscr{R}^L$ and $\mathscr{R} > \mathscr{R}^L$; the bifurcation is then termed two-sided.

increases. This would appear to be a contradiction of a theorem of Serrin (1960). They present results for the steady problem and make a comparison with the data of Taylor (1923) and the energy bounds of Serrin (1959). Their calculation is based, however, on an inner radius $R_1 = 3{\cdot}80$ cm and an outer radius $R_2 = 4{\cdot}035$ cm, while the results of Taylor and Serrin are for $R_1 = 3{\cdot}55$ cm and $R_2 = 4{\cdot}035$ cm. Re-evaluation of their criterion shows that it is particularly sensitive to gap width and is in agreement with neither the results of Serrin nor the data of Taylor.

In the following we examine the stability of circular Couette flow by the energy method. We shall restrict attention to the case when the outer cylinder is at rest and the inner has angular velocity given by (1). We derive criteria determining strong stability (Serrin 1959) but also a weaker concept of stability requiring only that disturbances should decay from cycle to cycle (Davis & von Kerczek 1973). This formulation allows the kinetic energy to increase during part of a period of oscillation of the primary flow, but requires that it must have decayed, at any instant, from the value observed $2\pi/\omega$ earlier. To distinguish this criterion from that of strong stability we shall refer to it as 'mean' energy theory. In § 2 we formulate the energy equations in the narrow-gap limit and describe the solution method. In § 3 we present the numerical results and compare the predictions with the linear theory of Riley & Laurence (1976). In § 4 we briefly summarize the results and make suggestions concerning further work.

2. Formulation of the stability problem

2.1. *Primary velocity field*

Suppose that a viscous incompressible fluid with kinematic viscosity ν is contained between infinitely long coaxial cylinders of radii R_1 and R_2 with $R_2 > R_1$. We shall assume that the gap width $d = R_2 - R_1$ is small compared with R_1, i.e. $\delta = d/R_1 \ll 1$. Terms of order δ will be neglected in the following analysis. Let (r', θ', z') be cylindrical polar co-ordinates with z' parallel to the generators. We take d, $R_1\bar{\Omega}$ and d^2/ν as the reference length, velocity and time, where $\bar{\Omega}$ will be taken to be equal to Ω_m for flows with non-zero mean and to be equal to Ω_p for flows with zero mean. We define dimensionless variables

$$x = (r' - R_1)/d, \quad z = z'/d, \quad \tau = \nu t'/d^2.$$

The primary flow $\mathbf{U}' = (0, R_1, \bar{\Omega} V, 0)$ is adequately represented in the limit $\delta \to 0$ by the solution of

$$\partial V/\partial \tau = \partial^2 V/\partial x^2,$$

$$V(0,\tau) = s + \bar{\epsilon}\cos\omega\tau, \quad V(1,\tau) = 0,$$

where

$$\omega = \omega' d^2/\nu, \quad s = \Omega_m/\bar{\Omega}, \quad \bar{\epsilon} = \Omega_p/\bar{\Omega}.$$

The solution may be written as

$$V(x,\tau) = s(1-x) + \bar{\epsilon}\{f_1(x)\cos\omega\tau + f_2(x)\sin\omega\tau\}, \tag{7}$$

where

$$f_1(x) = \{g_1(0)\, g_1(x) + g_2(0)\, g_2(x)\}/W(\gamma),$$

$$f_2(x) = \{g_2(0)\, g_2(x) - g_1(0)\, g_2(x)/W(\gamma),$$

$$g_1(x) = \sinh\bar{\lambda}(x)\cos\bar{\lambda}(x), \quad g_2(x) = \cosh\bar{\lambda}(x)\sin\bar{\lambda}(x),$$
$$\bar{\lambda}(x) = \gamma(1-x), \quad W(\gamma) = g_1^2(0)+g_2^2(0).$$

We refer to γ as the frequency; it is equal, however, to the inverse of the dimensionless Stokes-layer 'thickness'. We note for future discussion that at low frequencies

$$V(x,\tau) \sim (1-x)(s+\bar{\epsilon}\cos\omega\tau) \tag{8}$$

while at high frequencies

$$V(x,\tau) \sim s(1-x)+\bar{\epsilon}e^{-\gamma x}\cos(\omega\tau-\gamma x). \tag{9}$$

The complete profile (7) was used for all numerical calculations.

2.2. *Formulation of the stability problem*

Difference-motion equations. We suppose that the primary velocity and pressure fields $\{\mathbf{U}, \Pi\} = \{(0, V, 0), \Pi\}$ are disturbed to a new motion

$$\{(\tilde{u}, \tilde{v}+V, \tilde{w}), \tilde{p}+\Pi\}.$$

The dimensionless difference variables satisfy

$$\mathscr{R}^{-1}\,\partial\tilde{\mathbf{u}}/\partial\tau+\tilde{\mathbf{u}}\,.\,\nabla\tilde{\mathbf{u}}+\mathbf{U}\,.\,\nabla\tilde{\mathbf{u}}+\tilde{\mathbf{u}}\,.\,\nabla\mathbf{U} = -\nabla\tilde{p}+\mathscr{R}^{-1}\Delta\tilde{\mathbf{u}}, \tag{10a}$$

$$\nabla\,.\,\tilde{\mathbf{u}} = 0, \tag{10b}$$

$$\tilde{\mathbf{u}} = 0 \quad \text{on solid boundaries,} \tag{10c}$$

where $\mathscr{R} = (R_1\bar{\Omega}d)/\nu$ is the Reynolds number. The secondary motion is observed to have a Taylor-vortex structure (Donnelly 1964). We shall suppose disturbances to be axially periodic with dimensionless wavenumber α. Let $\mathscr{V}$ be the annular volume between the cylinders defined by the period of a vortex. By taking the scalar product of (10a) with $\tilde{\mathbf{u}}$, integrating over $\mathscr{V}$ and using conditions (10b,c) and the divergence theorem, the following energy evolution equation results:

$$d\mathscr{K}/d\tau = -(\mathscr{I}\mathscr{R}+\mathscr{D}), \tag{11}$$

where $$\mathscr{K} = \frac{1}{2}\int_{\mathscr{V}}\tilde{\mathbf{u}}\,.\,\tilde{\mathbf{u}}\,d\mathbf{x}, \quad \mathscr{I} = \int_{\mathscr{V}}\tilde{\mathbf{u}}\,.\,\mathbf{D}\,.\,\tilde{\mathbf{u}}\,d\mathbf{x}, \quad \mathscr{D} = \int_{\mathscr{V}}\nabla\tilde{\mathbf{u}}:\nabla\tilde{\mathbf{u}}\,d\mathbf{x}$$

and $\mathbf{D} = \frac{1}{2}\{\nabla\mathbf{U}+(\nabla\mathbf{U})^{\mathrm{T}}\}$ is the rate-of-deformation tensor.

Strong energy bounds. A motion is strongly globally stable in the mean if $\mathscr{K}(r) \to 0$ as $\tau \to \infty$ for $\mathscr{K}(0) < \infty$ and if $\mathscr{K}(\tau)$ decreases from the start.† Let

$$1/\tilde{\mathscr{R}}(\tau) = \max_{\mathscr{S}} -(\mathscr{I}/\mathscr{D}), \tag{12}$$

where $\mathscr{S}$ is a class of twice continuously differentiable solenoidal functions on $\mathscr{V}$ which vanish on solid boundaries. We write

$$\mathscr{R}^E = \min_{\tau\in[0,\,2\pi/\omega]} \tilde{\mathscr{R}}(\tau).$$

† There are disturbances which, although asymptotically stable, i.e. $\mathscr{K}(\tau) \to 0$ as $\tau \to \infty$, are persistent in the sense that $\mathscr{K}(\tau)$ may increase for a time. The requirement that this should not occur and that the motion be asymptotically stable is equivalent to demanding that $d\mathscr{K}/d\tau$ be negative definite.

Then if $\mathscr{R} < \mathscr{R}^E$ we may show (Joseph 1966) that

$$\mathscr{K}(\tau) \leqslant \mathscr{K}(0)\exp\{\xi^2(\mathscr{R}/\mathscr{R}^E - 1)\},$$

where ξ^2 is a positive constant depending only on the geometry. Hence the motion is strongly globally stable.

The Euler–Lagrange equations corresponding to (12) may be written as

$$\left.\begin{aligned} -\nabla\tilde{p} - \tilde{\mathscr{R}}(\tau)\,\mathbf{D}.\tilde{\mathbf{u}} + \Delta\tilde{\mathbf{u}} &= 0,\\ \nabla.\tilde{\mathbf{u}} &= 0,\\ \tilde{\mathbf{u}} = 0 \quad \text{on solid boundaries},& \end{aligned}\right\} \tag{13}$$

where $\tilde{p}(x,\theta,z;\tau)$ and $\tilde{\mathscr{R}}(\tau)$ are Lagrange multipliers. For fixed $\tau \in [0, 2\pi/\omega]$ we may determine $\tilde{\mathscr{R}}$ as the minimum positive eigenvalue of (13).

Mean energy bounds. A motion is globally asymptotically stable in the mean if $\mathscr{K}(\tau) \to 0$ as $\tau \to \infty$ for $\mathscr{K}(0) < \infty$. We define

$$e(\tau;\mathscr{R}) = \max_{\mathscr{S}} -\{(\mathscr{J}\mathscr{R} + \mathscr{D})/\mathscr{K}\}; \tag{14}$$

then from (11) and (14)

$$\ln\frac{\mathscr{K}(\tau)}{\mathscr{K}(0)} \leqslant \int_0^\tau e(\zeta;\mathscr{R})\,d\zeta.$$

It follows that a $2\pi/\omega$-periodic motion will be globally asymptotically stable in the mean if

$$\int_0^{2\pi/\omega} e(\zeta;\mathscr{R})\,d\zeta \leqslant 0 \tag{15}$$

since this implies that $\mathscr{K}(\tau + 2\pi/\omega) \leqslant \mathscr{K}(\tau)$ for all $\tau \in (0,\infty)$. The maximum value of $\mathscr{R}$, call it $\mathscr{R}^M$, for which (15) is satisfied if $\mathscr{R} \leqslant \mathscr{R}^M$ is termed the mean energy bound.

Problem (14) is equivalent to

$$\left.\begin{aligned} e(\tau;\mathscr{R})\,\tilde{\mathbf{u}} &= -\nabla\tilde{p} - \mathscr{R}\mathbf{D}.\tilde{\mathbf{u}} + \Delta\tilde{\mathbf{u}},\\ \nabla.\tilde{\mathbf{u}} &= 0,\\ \tilde{\mathbf{u}} = 0 \quad \text{on solid boundaries.}& \end{aligned}\right\} \tag{16}$$

For fixed $\mathscr{R}$ and τ, $e(\tau;\mathscr{R})$ may be found as the maximum eigenvalue of (16), and $\mathscr{R}^M$ evaluated by the requirement that (15) should hold with equality. We note that (13) becomes identical to (16) if we set $e(\tau;\mathscr{R}) = 0$.

Equations (13) and (16) bear a passing resemblance to the equations of linear stability theory. However, the transfer of energy between the primary and disturbance motion is described by terms of the form $\mathbf{D}.\tilde{\mathbf{u}}$ rather than

$$\tilde{\mathbf{u}}.\nabla\mathbf{U} + \mathbf{U}.\nabla\tilde{\mathbf{u}}.$$

Equations (13) and (16) are in addition self-adjoint. The solutions of (13) and (16) are, in general, only kinematically admissible solutions of the Navier–Stokes equations and not dynamically admissible, in contrast to the eigenfunctions of linear stability theory, which are both kinematically and dynamically admissible, at least in the limit of zero amplitude. It follows that $\mathscr{R}^L$ and $\mathscr{R}^E$ are equal only for very special motions, e.g. the Bénard problem, or for a more restrictive energy theory applied to a specific problem (e.g. Joseph & Hung 1971).

Non-axisymmetric disturbances have not been observed experimentally (Thompson 1968). Thus we shall look for axisymmetric solutions of (13) and (16) of the form

$$\tilde{u}(x,\theta,z;\tau) = u(x;\tau)\cos\alpha z, \quad \tilde{w}(x,\theta,z;\tau) = w(x;\tau)\sin\alpha z,$$

$$\tilde{v}(x,\theta,z;\tau) = v(x;\tau)\cos\alpha z, \quad \tilde{p}(x,\theta,z;\tau) = p(x;\tau)\cos\alpha z.$$

After some algebra, assuming $\delta \to 0$ and letting

$$\alpha v \to v, \tag{17}$$

equations (16) become

$$e\mathscr{L}u = \mathscr{L}^2 u + \tfrac{1}{2}\alpha\mathscr{R}(\partial V/\partial x)\,v, \tag{18a}$$

$$ev = \mathscr{L}v - \tfrac{1}{2}\alpha\mathscr{R}(\partial V/\partial x)\,u, \tag{18b}$$

$$u = v = \partial u/\partial x = 0 \quad \text{at} \quad x = 0, 1, \tag{18c}$$

where $\mathscr{L} = \partial^2/\partial x^2 - \alpha^2$. After identical manipulations, equations (13) are given by (18) with $e \equiv 0$. For modulation with non-zero mean we set $\overline{\Omega} = \Omega_m$; then

$$s = 1, \quad \bar{\epsilon} = \epsilon, \quad \mathscr{R} \equiv \mathscr{R}_m = (R_1\Omega_m d)/\nu.$$

For modulation with zero mean we set $\overline{\Omega} = \Omega_p$; then

$$s = 0, \quad \epsilon = 1{\cdot}0, \quad \mathscr{R} \equiv \mathscr{R}_p = (R_1\Omega_p d)/\nu.$$

The equations (18) used in our energy analysis result from applying the narrow-gap approximation to the wide-gap Euler–Lagrange equations (13) and (16). The characteristic parameter arising from this procedure is a Reynolds number $\mathscr{R}$ rather than the more familiar Taylor number $2\delta\mathscr{R}^2$. The latter may be calculated from an energy analysis only if the wide-gap problem is solved, or if a narrow-gap energy formulation is derived which requires that the initial energy is restricted in magnitude. Such an energy principle could be derived from the narrow-gap equations as formulated by Stuart (1958).

The results we shall present will not be global since we are assuming axisymmetric solutions and hence our bounds do not necessarily guarantee stability against non-axisymmetric disturbances. However our results are in a sense optimal; stability cannot be guaranteed against axisymmetric disturbances at a higher Reynolds number without restricting the initial energy of such disturbances (cf. Joseph & Hung 1971).

2.3. *Solution techniques*

Strong energy bounds. We solve the problem by Galerkin's method. The solution of (18) (with $e = 0$) is expanded in a complete set of functions satisfying the boundary conditions (18c):

$$u = \sum_{n=1}^{N} a_n(\tau)\,\phi_n, \quad v = \sum_{n=1}^{\infty} b_n(\tau)\,\tilde{\phi}_n. \tag{19a,b}$$

The functions $\{\phi_n\}$ and $\{\tilde{\phi}_n\}$ are defined in the appendix. Convergence is assumed as $N \to \infty$. After forcing the truncation error to be orthogonal to the expansion functions in the normal way (see appendix), we derive the eigenvalue problem

$$|\mathbf{M} - \beta \mathbf{I}| = 0, \tag{20}$$

where $|\beta| = |2/\alpha\mathscr{R}|$, $\mathbf{M}$ is a $2N \times 2N$ matrix defined in the appendix and $\mathbf{I}$ is the $2N \times 2N$ identity matrix. For fixed N, γ, ϵ, τ and α let $\hat{\beta}(\alpha, \tau)$ be the eigenvalue of (20) largest in absolute value and let

$$\tilde{\mathscr{R}}(\alpha, \tau) = 2/\alpha|\hat{\beta}(\alpha, \tau)|.$$

Then
$$\mathscr{R}^E = \min_{\substack{\alpha,\ \alpha>0 \\ \tau,\ \tau \in [0,\ 2\pi/\omega]}} \tilde{\mathscr{R}}(\alpha, \tau). \tag{21}$$

The two-dimensional search (21) was performed by means of two one-dimensional searches. For α fixed, $\tilde{\mathscr{R}}(\alpha, \tau)$ was minimized as a function of τ using an accelerated search with quadratic fitting as a predictor (Jacoby, Kowalik & Pizzo 1972, p. 69). The search was terminated when the increment size $\Delta\tau$ before the next prediction satisfied $|\Delta\tau| \leqslant 5 \times 10^{-3}(2\pi/\omega)$. With τ fixed at this intermediate optimal value τ^*, $\mathscr{R}(\alpha, \tau^*)$ was minimized as a function of α using the same technique until the increment size $\Delta\alpha$ before the next prediction satisfied $|\Delta\alpha| < 5 \times 10^{-3}$. The procedure was then repeated until the critical value $\mathscr{R}_e^E(N)$ was determined with the accuracy required. The restrictions on $|\Delta\alpha|$ and $|\Delta\tau|$ are in fact sufficiently stringent that $\mathscr{R}_e^E(N)$ is found accurate to three decimal places.

Mean energy bounds. We solve the problem by Galerkin's method using the same expansion functions as before. After the usual manipulations, (18) is found to be equivalent to the eigenvalue problem

$$|\mathbf{H} - e\mathbf{I}| = 0, \tag{22}$$

where $\mathbf{H}$ is a $2N \times 2N$ matrix defined in the appendix. For α and $\mathscr{R}$ fixed and arbitrary τ in $[0, 2\pi/\omega]$ we may find $e(\tau; \mathscr{R}, \alpha)$ as the maximum eigenvalue of (22).

Let
$$E(\mathscr{R}, \alpha) = \int_0^{2\pi/\omega} e(\zeta; \mathscr{R}, \alpha)\, d\zeta. \tag{23}$$

Critical values $\mathscr{R}_c$ and α_c of the Reynolds number and wavenumber are found from the requirement that

$$|E(\mathscr{R}^*, \alpha)| \leqslant 10^{-4}, \quad \mathscr{R}_c = \mathscr{R}_c^* = \min_{\alpha} \mathscr{R}^*(\alpha). \tag{24a, b}$$

The integration in (23) was performed using sixteen-point Gaussian quadrature. For fixed N, γ, ϵ and α, a value of $\mathscr{R}^*$, the Reynolds number satisfying (24a), was found using the secant method (Ralston 1965, p. 323). $E(\mathscr{R}^*, \alpha)$ was then maximized as a function of α again using the technique described by Jacoby *et al.* The search was ended when $|\Delta\alpha| \leqslant 5 \times 10^{-3}$. The procedure was then repeated until (24a, b) were satisfied with the accuracy required. The criteria are sufficiently stringent that $\mathscr{R}_c^M$ is found accurate to three decimal places.

The matrix inversions required to form (20) and (22) and the evaluation of the eigenvalues were carried out using routines provided by the University of

Massachusetts Computing Center. The inversion program uses Gauss–Jordan elimination with complete pivoting. The eigenvalue routine is based on the QR algorithm (Francis 1961; Wilkinson 1965). In all cases it was found that the trace of **M** (or **H**) was equal to the sum of the eigenvalues to at least nine decimal places.

In the following, superscripts E and M will refer to strong and mean energy bounds respectively. $\mathscr{R}_m$ denotes the Reynolds number $\mathscr{R}_m = (R_1 \Omega_m d)/\nu$, while $\mathscr{R}_p = (R_1 \Omega_p d)/\nu$. A subscript zero will denote the critical conditions for steady flow, viz. $\mathscr{R}_0$ and α_0, where $\mathscr{R}_0 = (R_1 \Omega_0 d)/\nu$. A superscript L will denote values of critical parameters from linear theory.

3. Results and discussion

3.1. *Precision of the numerical procedure*

An analysis of the steady problem ($\epsilon = 0$) was used as an initial test of the calculation procedure. In the case $\epsilon = 0$, it is clear that the strong and mean energy bounds should be equal. Using either technique, it was found for $N = 3$ that $\alpha_0 = 3{\cdot}116$ and $\mathscr{R}_0 = 82{\cdot}669$ and for $N = 5$ that $\alpha_0 = 3{\cdot}116$ and $\mathscr{R}_0 = 82{\cdot}650$. This problem proves to be mathematically equivalent to the Bénard problem with rigid conducting boundaries, for which Chandrasekhar (1961, p. 38) gives (in our notation) $\alpha_0 = 3{\cdot}117$ and $\mathscr{R}_0 = 82{\cdot}650$.† It can be seen that the results are in remarkably close agreement.

For the steady problem the answer found with $N = 5$ may be considered essentially correct. For modulated flow, however, the oscillatory Stokes layer is considerably more complex than the mean shear, and hence the modulated problem requires more terms in the expansion for comparable accuracy whenever ϵ and γ are large. All the results presented are for $N = 5$, however, as larger values of N require excessive computing time and were only used for convergence checks. $\mathscr{R}$ was found to be a monotonically decreasing function of N. The convergence of the wavenumber was oscillatory on occasion. The maximum error is estimated to be $0{\cdot}03\,\%$ for $\epsilon = 0{\cdot}1$, about $1{\cdot}5\,\%$ for $\epsilon = 2{\cdot}0$ and to be $2{\cdot}5\,\%$ for modulation with zero mean.

We present values of critical parameters only for $0 < \gamma \leqslant 10$, as the linear results of Riley & Laurence suggest that this is the frequency range of greatest interest. For modulated flow with non-zero mean the Stokes layer is confined to a region close to the inner cylinder as $\gamma \to \infty$ [see (9)], and for $\gamma > 10{\cdot}0$ the linear stability of the system is determined by the mean shear.

Linear theory derives critical bounds in terms of the parameter $\mathscr{R}^L\sqrt{\delta}$. For comparison of the predictions of linear and energy analyses we have (arbitrarily) chosen $\delta = 0{\cdot}0444$, corresponding to the experiments of Thompson (1968).

† If the primary flow is steady, (15) is satisfied only if $e(\tau;\mathscr{R}) \leqslant 0$. Then the strong and mean energy bounds are both given by (18) with $e \equiv 0$. Since $\partial V/\partial x = -1$ for steady flow, (18) is then equivalent to the Bénard problem for rigid boundaries if we associate u with the vertical component of velocity, v with the temperature and $\mathscr{R}$ with $2\sqrt{Ra}$, where Ra is the Rayleigh number.

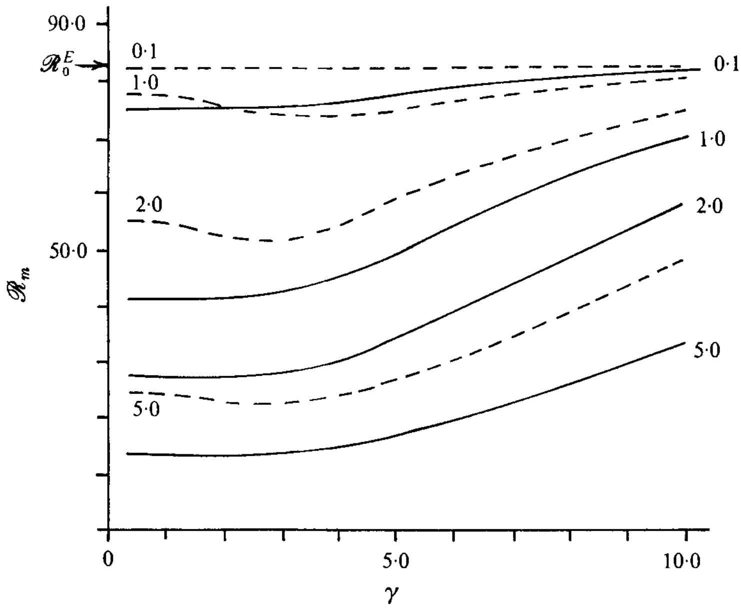


FIGURE 1. Critical Reynolds numbers $\mathscr{R}_m$ from strong and mean energy theory with the amplitude ratio ϵ as a parameter. ——, $\mathscr{R}_m^E$; – – – –, $\mathscr{R}_m^M$.

3.2. *Modulated flows with non-zero mean*

In the limit $\gamma \to 0$, the primary flow is given with good accuracy by (8).† If $\mathscr{R}_0^E$ is the critical Reynolds number for strong stability of steady flow, it is clear that modulated flow with non-zero mean will be strongly stable if

$$|\mathscr{R}_m^E(1+\epsilon\cos\omega\tau)| \leqslant \mathscr{R}_0^E,$$

i.e. if
$$\mathscr{R}_m^E = \mathscr{R}_0^E/(1+\epsilon), \tag{25}$$

and modulated flow with zero mean will be strongly stable if

$$|\mathscr{R}_p^E\cos\omega\tau| \leqslant \mathscr{R}_0^E,$$

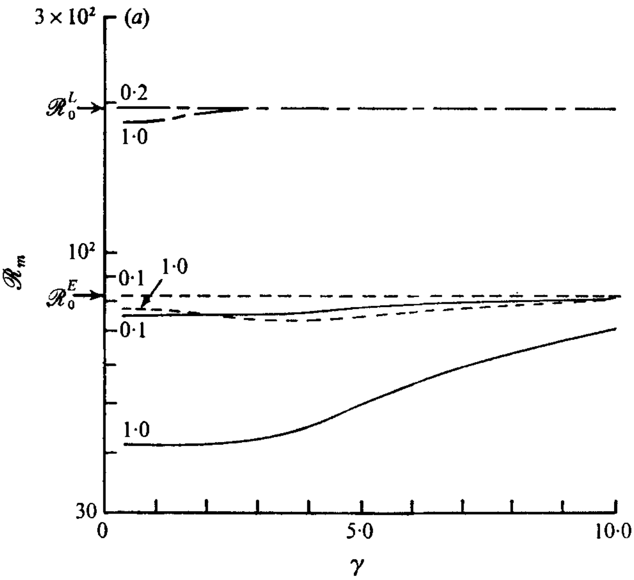
i.e. if
$$\mathscr{R}_p^E = \mathscr{R}_0^E. \tag{26}$$

Thus it is evident that modulation is destabilizing in the low frequency limit. Calculation with the complete profile (6) shows that (25) and (26) hold with good accuracy for $\gamma \leqslant 2{\cdot}0$. Evidently, (25) and (26) are still valid if the narrow-gap approximation is not made, but with $\mathscr{R}_0^E$ replaced with its wide-gap value.

† The correct expression is (cf. Hall 1975)

$$V(x,\tau) = (1-x)+\epsilon\{(1-x)\cos\omega\tau+\tfrac{1}{3}\gamma^2(x^3+2x-3x^2)\sin\omega\tau\}.$$

Evidently (8) is not uniformly valid in τ, in particular when $\cos\omega\tau = 0$. However, when $\gamma \to 0$ the factor premultiplying $\sin\omega\tau$ is very much less than $1-x$. The strong energy bound is determined by $V(x,\tau)$ at a single value of τ, for which $\cos\omega\tau \neq 0$. Hence the results are dominated by the mean shear and the term involving $\cos\omega\tau$. Equation (8) is used here to illustrate the expected behaviour of the critical parameters, behaviour which is borne out by the numerical results found using the complete profile (7).



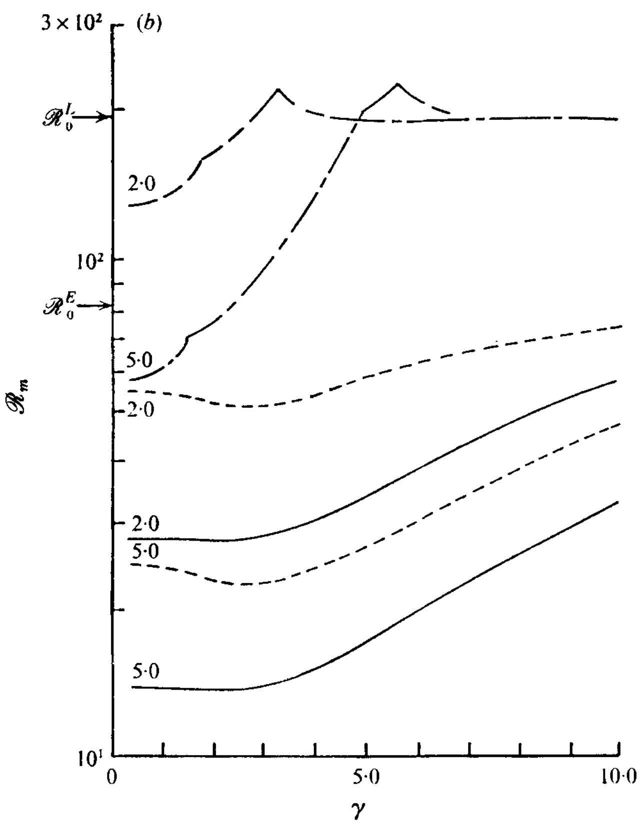


FIGURE 2. Comparison of the predictions of critical Reynolds numbers from the linear and energy analyses. (a) $\epsilon = 0{\cdot}1,\ 1{\cdot}0$. (b) $\epsilon = 2{\cdot}0,\ 5{\cdot}0$. $\mathscr{R}_0^L$ and $\mathscr{R}_0^E$ are the critical values for steady flow from the linear and energy analyses. ——, $\mathscr{R}_m^E$; – – – –, $\mathscr{R}_m^M$; — — —, $\mathscr{R}_m^L$.

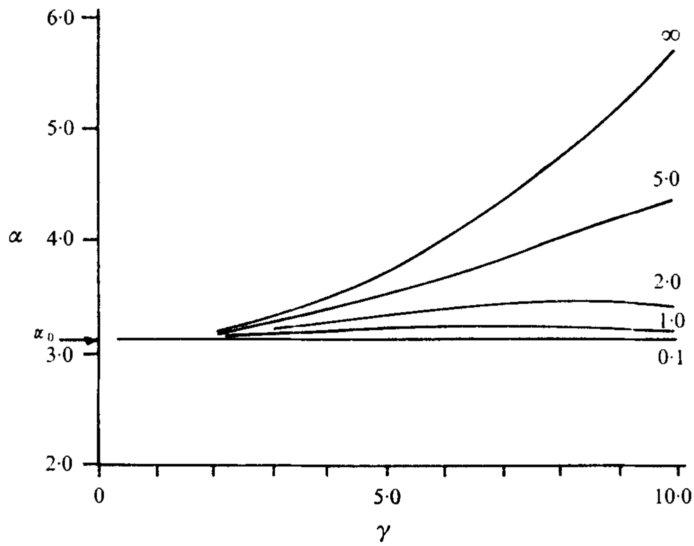


FIGURE 3. Critical wavenumber from the strong energy analysis with the amplitude ratio as a parameter. α_0^E is the critical value for steady flow.

The critical Reynolds numbers from both the strong and the mean energy method are plotted in figure 1 as a function of frequency, with the amplitude ratio ϵ as a parameter. For $\epsilon \leqslant 2{\cdot}0$ the strong energy bound increases monotonically from $\mathscr{R}_0^E/(1+\epsilon)$ as the frequency increases from zero, approaching $\mathscr{R}_0^E$ asymptotically from below as the frequency becomes large. For $\epsilon = 5{\cdot}0$ the behaviour is modified only by $\mathscr{R}_m^E$ passing through a slight maximum and then a minimum. The mean energy bounds $\mathscr{R}_m^M$ show a similar overall trend, approaching $\mathscr{R}_0^E$ from below as the frequency becomes large. The minimum is now, however, evident for $\epsilon \geqslant 1{\cdot}0$ and is more pronounced; the maximum is absent. The minimum occurs approximately at $\gamma \approx 3{\cdot}0$, and is largely independent of the amplitude ratio. It is due to the shape of the velocity profile; the importance of in-phase and out-of-phase components of the Stokes layer varies as the frequency increases. At low frequencies the in-phase component dominates, and as the frequency increases the components may reinforce or oppose one another (and the mean shear, if present) to differing degrees. The mean energy criterion enlarges the region of stability quite significantly.

In figures 2(a) and (b) we compare the linear predictions with those of both energy criteria. For $\epsilon \geqslant 2{\cdot}0$ (figure 2b), the curves of critical Reynolds number from linear theory exhibit cusps. This is associated with a change in linear response and discontinuous jumps in the critical wavenumber–frequency curves (the interested reader is referred to Riley & Laurence 1976). We are not primarily interested in the details of the linear response, but only in an overall comparison with the predictions of the energy methods. Energy theory suggests that modulation will destabilize the flow, but decreasingly as the frequency increases. This is also the overall trend of the linear predictions, though for $\epsilon > 1{\cdot}0$ weak stabilization is possible at intermediate frequencies. The linear and mean energy

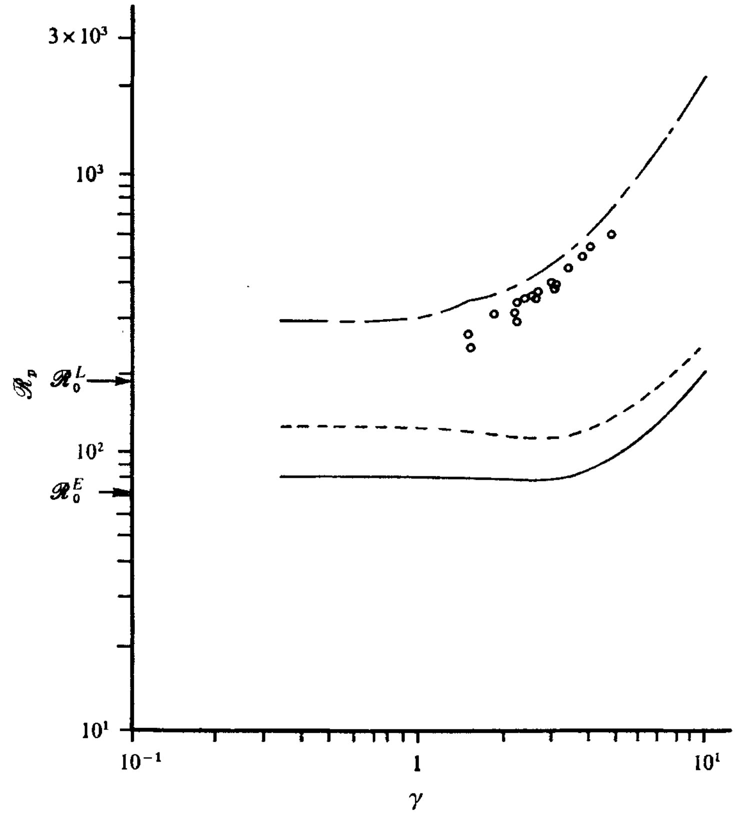


FIGURE 4. Critical Reynolds number $\mathscr{R}_p$ for modulated flow with zero mean compared with the data of Thompson (1968) for $\delta = 0{\cdot}0444$, $R_1 = 6{\cdot}027$ cm. $\mathscr{R}_0^L$ and $\mathscr{R}_0^E$ are the critical values for steady flow from linear and energy theory respectively. ——, $\mathscr{R}_p^E$; - - - -, $\mathscr{R}_p^M$; — — —, $\mathscr{R}_p^L$.

analyses suggest for $\epsilon = 0{\cdot}1$ that the critical parameters will be largely independent of frequency. Both these criteria define instability as occurring if there is disturbance growth from cycle to cycle, and both suggest that modulation at such low amplitude ratios is too weak to affect stability. One suspects that this will be a feature of experiments for frequencies sufficiently high that a periodicity criterion is a relevant one.

Figure 3 is a plot of the critical wavenumber α^E from strong energy theory as a function of frequency ($\epsilon = \infty$ corresponding to modulation with zero mean). The predictions of mean energy theory are similar and are not reproduced here. For flows with non-zero mean the wavenumber initially increases with frequency as the thickness of the Stokes layer decreases. Eventually, as the layer becomes increasingly narrow, the ability of the modulation to affect stability lessens. Thus the wavenumber passes through a maximum and decreases towards α_0 for $\gamma \rightarrow \infty$, as the instability becomes primarily associated with the mean shear. The stronger the modulation, the thinner the Stokes layer may become before this stage is reached. Thus for $\epsilon = 5{\cdot}0$ no maximum is evident for $\gamma \leqslant 10{\cdot}0$. The predictions of linear theory are not presented since the wavenumber changes rapidly and discontinuously as the linear response varies. However, for $\epsilon \leqslant 1{\cdot}0$, and for $\epsilon > 1{\cdot}0$ when the frequency is sufficiently high that instability is dominated by the mean rotation, the behaviour is similar. Linear and mean energy theory predict that at low frequencies the wavenumber should be less than the value for steady flow. The dependence of the reduction upon the amplitude ratio is similar for both.

3.3. *Modulated flow with zero mean*

Figure 4 compares the results of the linear and energy methods with the data of Thompson (1968). For $\gamma > 2{\cdot}0$ the linear predictions and experiment seem to be in reasonable agreement, but for $\gamma < 2{\cdot}0$ agreement is poor. The energy methods as formulated are evidently too conservative to be more reliable predictive techniques at low frequencies.

As the Stokes layer becomes confined to the inner cylinder as the frequency increases, we expect the critical parameters to become independent of gap width. It follows that both the linear and the energy analyses will predict that as γ becomes large $\alpha \sim \gamma$. However, linear theory suggests

$$\mathscr{R}_p^L\surd\delta \sim \gamma^{\frac{3}{2}} \quad \text{or} \quad \hat{\mathscr{R}}^L \sim \hat{\gamma}^{\frac{3}{2}},$$

where

$$\hat{\mathscr{R}} = (R_1\Omega_p)\,R_1/\nu, \quad \hat{\gamma} = (\omega' R_1^2/2\nu)^{\frac{1}{2}},$$

while energy theory predicts

$$\mathscr{R}_p^E \sim \gamma, \quad \mathscr{R}_p^M \sim \gamma$$

or

$$\hat{\mathscr{R}}^E \sim \hat{\gamma}, \quad \hat{\mathscr{R}}^M \sim \hat{\gamma}.$$

Numerically we find

$$\alpha = 0{\cdot}85\gamma, \quad \hat{\mathscr{R}} \sim 15{\cdot}3\hat{\gamma}^{\frac{3}{2}},$$

$$\alpha^E = 0{\cdot}57\gamma, \quad \hat{\mathscr{R}}^E \sim 19{\cdot}5\hat{\gamma},$$

$$\alpha^E = 0{\cdot}57\gamma, \quad \hat{\mathscr{R}}^M \sim 24{\cdot}4\hat{\gamma}.$$

The numerical values of the constants are reliable for the linear predictions for $\gamma \geqslant 8{\cdot}0$. For the energy analyses the values should be considered approximate. Hence the dependence of critical Reynolds numbers upon frequency is altogether different in the high frequency limit. We note that this observation is independent of the narrow-gap approximation.

4. Conclusions

At low frequencies modulated flow with non-zero mean becomes unstable when (4) holds, i.e.

$$\mathscr{R}_m = \mathscr{R}_0/(1+\epsilon), \tag{27}$$

while modulated flow with zero mean becomes unstable whenever (5) holds, i.e.

$$\mathscr{R}_p = \mathscr{R}_0. \tag{28}$$

For the steady problem linear theory is an accurate estimate of the onset of instability; i.e. $\mathscr{R}_0 = \mathscr{R}_0^L$. It follows from (25) and (26) that strong energy theory will provide an accurate estimate of the onset of instability for the modulated flow at low frequencies if $\mathscr{R}_0^E \approx \mathscr{R}_0^L$. From the calculations we have presented, this is evidently not the case: our criterion for global stability of axisymmetric disturbances is too stringent, and $\mathscr{R}_0^E$ is significantly less than $\mathscr{R}_0^L$.

Joseph & Hung (1971) have, however, presented a modified energy theory for circular Couette flow, which requires the motion to be stable in the sense that the kinetic energy of disturbances of finite but limited size decays from the start. In addition, the threshold amplitude may be calculated. On the basis of this weaker concept of stability the 'modified' energy bound is found to be very close to the linear value. The arguments used to derive (25) and (26) apply equally well to this modified theory, and we may conjecture, therefore, that (27) and (28) would be in exact agreement with the results of this modified energy theory when applied to the modulated problem.

Linear theory provides a sufficient condition for instability. The energy method provides a sufficient condition for stability (though as formulated here only for a certain class of disturbances). The techniques are, therefore, complementary. The size of the region between the two bounds provides some indication of the likelihood of subcritical instability and whether this is most probably the dominant mode of instability observed experimentally. For the problem considered here the region open to possible subcritical instability, though not large in comparison with that in situations such as plane Couette flow, is still reasonably extensive in absolute terms. We have remarked, throughout our discussion, that the overall trends of $\mathscr{R}^L(\gamma)$, $\mathscr{R}^E(\gamma)$ and $\mathscr{R}^M(\gamma)$ are similar. It seems unlikely that the predictions of the 'modified' energy theory would depart from these overall trends. Thus the region open to subcritical instability would be reasonably small. Relation (5) is consistent with linear theory at higher frequencies. We might expect in these circumstances that nonlinear perturbation theory based on the linear solution would give an accurate description of the secondary motion. At low frequencies the instability observed is subcritical and a mathematical description of such secondary motion would be considerably more complex. Whenever the onset of instability is given by (27) and (28) the disturbance growth observed is the result of instability of an effectively steady, quasi-static profile. If ϵ is large it may be possible that the growth observed during one cycle is entirely independent of that observed in successive cycles.

It would appear worthwhile to apply the modified energy theory to the modulated problem for the entire range of frequencies of interest. This should certainly provide accurate bounds at low frequencies, and narrow the region open to subcritical growth, i.e. the region between certain instability and conditional nonlinear stability.

Appendix

The functions ϕ_n and $\tilde{\phi}_n$ are defined by

$$\phi_n(x) = x^2(1-x)^2 \mathscr{J}_{n-1}(x), \quad \tilde{\phi}_n(x) = x(1-x)\, \tilde{\mathscr{J}}_{n-1}(x),$$

where $\mathscr{J}_n$ and $\tilde{\mathscr{J}}_n$ are Jacobi polynomials of order n and are normalized by the requirements

$$\int_0^1 x^2(1-x)^2 \mathscr{J}_n \mathscr{J}_m \, dx = \delta_{nm} \frac{n!\,4}{(5+2n)(4+n)!},$$

$$\int_0^1 x(1-x)\, \tilde{\mathscr{J}}_n \tilde{\mathscr{J}}_m \, dx = \delta_{nm} \frac{n!}{(2+n)!\,(3+2n)}$$

(see Morse & Feshbach 1953, p. 780). The sets $\{\phi_n\}_{n=1}^{\infty}$ and $\{\tilde{\phi}_n\}_{n=1}^{\infty}$ may be shown to be complete and minimal in $L^2[0, 1]$ (Riley 1975; see discussion below).

If the approximate solutions

$$u_N = \sum_{n=1}^{N} a_n \phi_n, \quad v_N = \sum_{n=1}^{N} b_n \tilde{\phi}_n$$

are substituted in ($18a, b$) a truncation error results:

$$-e\mathscr{L}u_N + \mathscr{L}^2 u_N + \tfrac{1}{2}\alpha\mathscr{R}(\partial V/\partial x)\, v_N = \epsilon_N, \qquad (\text{A}\,1a)$$

$$e v_N - \mathscr{L} v_N + \tfrac{1}{2}\alpha\mathscr{R}(\partial V/\partial x)\, u_N = \tilde{\epsilon}_N. \qquad (\text{A}\,1b)$$

The errors ϵ_N and $\tilde{\epsilon}_N$ are made orthogonal to the expansion functions ϕ_n and $\tilde{\phi}_n$ respectively for $n = 1, 2, \ldots, N$. We define

$$A_{ij} = -\int_0^1 \phi_i \mathscr{L} \phi_j \, dx, \quad \tilde{A}_{ij} = \int_0^1 \tilde{\phi}_i \tilde{\phi}_j \, dx,$$

$$B_{ij} = \int_0^1 \phi_i \mathscr{L}^2 \phi_j \, dx, \quad \tilde{B}_{ij} = -\int_0^1 \tilde{\phi}_i \mathscr{L} \tilde{\phi}_j \, dx,$$

$$C_{ij} = \int_0^1 \phi_i \left(\frac{\partial V}{\partial x}\right) \tilde{\phi}_j \, dx.$$

After the integrations have been performed (A 1) becomes

$$\left[e \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \tilde{\mathbf{A}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} & \tfrac{1}{2}\alpha\mathscr{R}\mathbf{C} \\ \tfrac{1}{2}\alpha\mathscr{R}\mathbf{C}^{\mathrm{T}} & \tilde{\mathbf{B}} \end{bmatrix} \right] \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = 0. \qquad (\text{A}\,2)$$

To calculate strong energy bounds we note that $e = 0$, and solve

$$\begin{vmatrix} \mathbf{B} & \tfrac{1}{2}\alpha\mathscr{R}\mathbf{C} \\ \tfrac{1}{2}\alpha\mathscr{R}\mathbf{C}^{\mathrm{T}} & \tilde{\mathbf{B}} \end{vmatrix} = 0. \qquad (\text{A}\,3)$$

If $\mathscr{R}$ is fixed and α satisfies (A 3) it follows from (17) that $-\alpha$ is also a solution. Hence the roots of (A 3) are of the form $\pm\left|\tfrac{1}{2}\alpha\mathscr{R}\right|$. Let $\beta = -2/\alpha\mathscr{R}$, then since the matrices $\mathbf{B}$ and $\tilde{\mathbf{B}}$ are positive definite and invertible, (A 3) is equivalent to

$$|\mathbf{M} - \beta\mathbf{I}| = 0,$$

where
$$\mathbf{M} = \begin{pmatrix} 0 & \mathbf{B}^{-1}\mathbf{C} \\ \tilde{\mathbf{B}}^{-1}\mathbf{C}^{\mathrm{T}} & 0 \end{pmatrix}.$$

To evaluate mean energy bounds, we note that $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are positive definite and hence invertible. Thus $e(\tau;\mathscr{R},\alpha)$ may be found as the maximum eigenvalue of

$$|\mathbf{H}-e\mathbf{I}| = 0,$$

where
$$\mathbf{H} = -\begin{pmatrix} \mathbf{A}^{-1}\mathbf{B} & \tfrac{1}{2}\alpha\mathscr{R}\mathbf{A}^{-1}\mathbf{C} \\ \tfrac{1}{2}\alpha\mathscr{R}\tilde{\mathbf{A}}^{-1}\mathbf{C}^{\mathrm{T}} & \tilde{\mathbf{A}}^{-1}\tilde{\mathbf{B}} \end{pmatrix}.$$

A sequence is minimal if the sequence formed from the original by deletion of one element no longer spans the original subspace. This condition on the expansion functions is necessary but not sufficient for a Galerkin type-procedure to be stable (Mikhlin 1971, p. 132). It is a requirement that the expansion functions be not too linearly dependent. Any choice of expansion functions should at least have this property. Functions of the form $x^2(1-x)^2x^k$ and $x(1-x)x^k$, while satisfying the boundary conditions required here, are not minimal and so are not used.

We could use the Chandrasekhar–Reid functions discussed by Chandrasekhar (1961, p. 634). However, one of our objectives was to evaluate the use of expansion functions of the type ϕ_n and $\tilde{\phi}_n$. It is possible (Riley 1975) to prove the general result that if $\{\psi_n\}$ is a complete orthogonal sequence in the space $L^2_w[a,b]$ with inner product defined by a continuous non-negative weighting function $w(x)$ and if $\{\psi_n\}$ is contained in $L^2[0,1]$, then the sequences $\{w(x)\,\psi_n(x)\}$ and $\{\psi_n(x)\}$ form a complete biorthogonal sequence in $L^2[0,1]$ and are both minimal. Since the properties of many orthogonal polynomials are well documented and the respective weighting functions may satisfy the required boundary conditions this result provides a convenient technique for constructing complete sets of functions satisfying prescribed boundary conditions for use in Galerkin's method.

Our numerical experiments showed $\{\phi_n\}$ and $\{\tilde{\phi}_n\}$ to provide a stable numerical scheme. Convergence (at least for the steady problem) appears to be slower than that obtained using the Chandrasekhar–Reid functions, presumably because they are similar to the actual eigenfunctions. For more general problems, Orszag & Israeli (1974) do not recommend the Chandrasekhar–Reid functions. Their extension to situations satisfying more than four boundary conditions, should this ever be required, would be tedious.

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